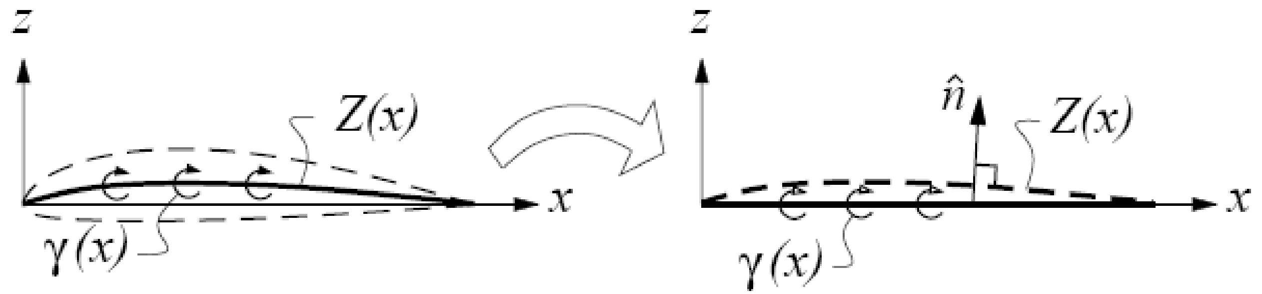
< 4.7 Classical Thin Airfoil Theory >

❖ The Symmetric Airfoil

* Assumptions

i) The camber line is one of the streamlines
ii) Small maximum camber and thickness relative to the chord
iii) Small angle of attack



* Purposes

i) Find γ(s)
ii) Use Kutta-Joukowski theorem, L′=ρVΓ

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❖ The Symmetric Airfoil

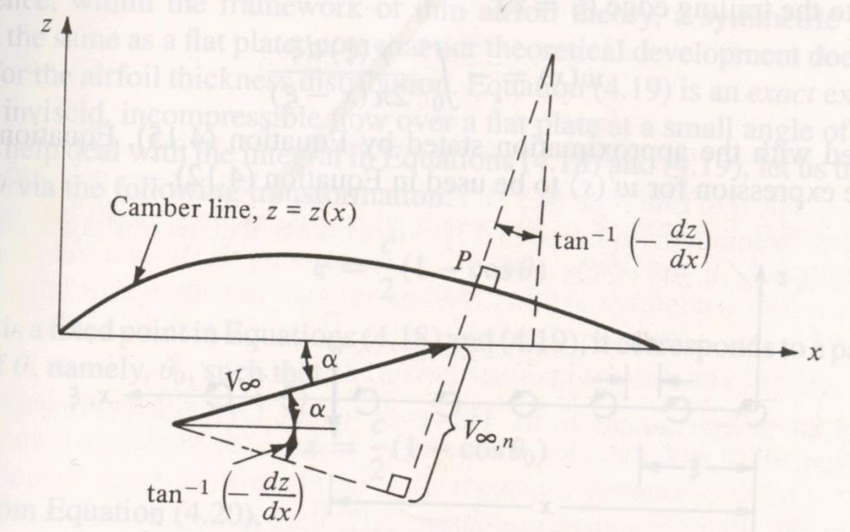

Figure 4.23 Determination of the component of freestream velocity normal to the camber line.

*** The component of free-stream velocity normal to the mean camber line at P**

➔ $$V_{\infty,n} = V_\infty \sin\left(\alpha + \tan^{-1}\left(-\frac{dz}{dx}\right)\right)$$

From small angle assumption

➔ $$V_{\infty,n} = V_\infty\left(\alpha - \frac{dz}{dx}\right)$$

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❖ The Symmetric Airfoil

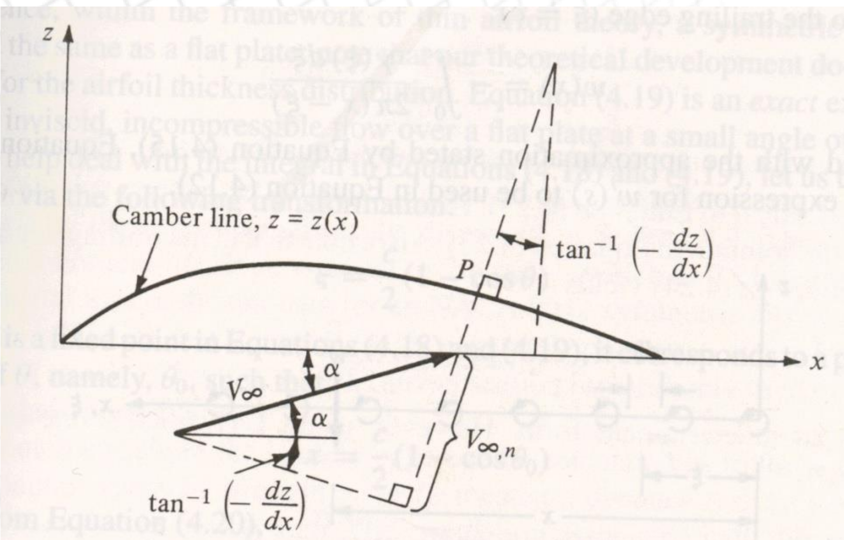

Figure 4.23 Determination of the component of freestream velocity normal to the camber line.

*** If the airfoil is thin,** $w'(s) \approx w(x)$

$w'(s)$: velocity normal to the **camber line** induced by the vortex sheet

$w(x)$: velocity normal to the **chord line** induced by the vortex sheet

*** The velocity at point x by the elemental vortex at point ξ**

$$\Rightarrow dw = -\frac{\gamma(\xi)d\xi}{2\pi(x-\xi)}$$

*** The velocity at point x by all the elemental vortices along the chord line**

$$\Rightarrow w(x) = -\int_0^c \frac{\gamma(\xi)d\xi}{2\pi(x-\xi)}$$

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❖ The Symmetric Airfoil

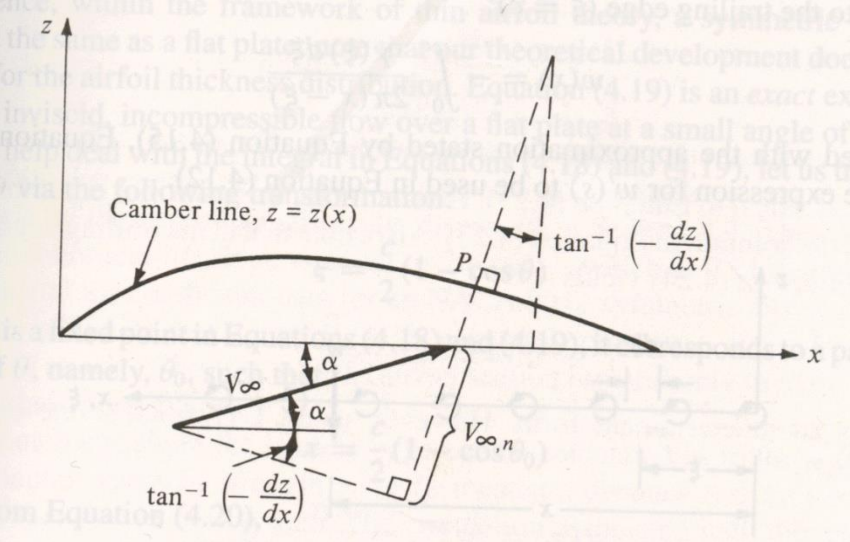

Figure 4.23 Determination of the component of freestream velocity normal to the camber line.

*** The sum of the velocity components normal to the surface at all point along the vortex sheet is zero**

➔ $V_{\infty,n} + w(x) = 0$

➔ $V_\infty\left(\alpha - \frac{dz}{dx}\right) - \int_0^c \frac{\gamma(\xi)d\xi}{2\pi(x-\xi)} = 0$

➔ $$\frac{1}{2\pi}\int_0^c \frac{\gamma(\xi)d\xi}{(x-\xi)} = V_\infty\left(\alpha - \frac{dz}{dx}\right)$$

↳ The fundamental equation of thin airfoil theory

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❖ The Symmetric Airfoil

* Sysmmetric airfoil ➔ no camber,

$$\frac{1}{2\pi}\int_0^c \frac{\gamma(\xi)d\xi}{(x-\xi)} = V_\infty\left(\alpha - \frac{dz}{dx}\right) \quad \xrightarrow{\frac{dz}{dx}=0} \quad \frac{1}{2\pi}\int_0^c \frac{\gamma(\xi)d\xi}{x-\xi} = V_\infty \alpha$$

* Transform variable ξ into θ

$$\xi = \frac{c}{2}(1-\cos\theta) \begin{cases} \theta = 0, \ \xi = 0 \\ \theta = \pi, \ \xi = c \end{cases}, \quad x = \frac{c}{2}(1-\cos\theta_0), \quad d\xi = \frac{c}{2} sin\theta \, d\theta$$

$$\frac{1}{2\pi}\int_0^c \frac{\gamma(\xi)d\xi}{x-\xi} = V_\infty \alpha \Rightarrow \frac{1}{2\pi}\int_o^\pi \frac{\gamma(\theta)\sin\theta \, d\theta}{\cos\theta - \cos\theta_0} = V_\infty \alpha \Rightarrow \gamma(\theta) = 2\alpha V_\infty\left(\frac{1+\cos\theta}{\sin\theta}\right)$$

$$\left(\int_o^\pi \frac{n cos\theta}{\cos\theta - \cos\theta_0} d\theta = \frac{\pi \sin n\theta_0}{\sin\theta_0}\right)$$

< 4.7 Classical Thin Airfoil Theory >

❖ The Symmetric Airfoil

$$\gamma(\theta) = 2\alpha V_\infty \left(\frac{1+\cos\theta}{\sin\theta}\right)$$

* Check Kutta condition

$$\lim_{\theta \to \pi} \gamma(\theta) = 2\alpha V_\infty \frac{0}{0}$$ ← Indeterminant form

By L'Hospital's rule

$$\rightarrow \gamma(\theta) = \lim_{\theta \to \pi} 2\alpha V_\infty \left(\frac{-\sin\theta}{\cos\theta}\right)_{\theta=\pi} = 2\alpha V_\infty (0) = 0$$

< 4.7 Classical Thin Airfoil Theory >

❖ The Symmetric Airfoil

* Since we get $\gamma(\theta)$, now calculate Γ, L

➔ $$\Gamma = \int_o^c \gamma(\xi) d\xi = \int_0^\pi \frac{c}{2}\gamma(\theta)\sin\theta\, d\theta$$ ➔ $$\Gamma = \alpha c V_\infty \int_o^\pi (1+\cos\theta) d\theta = \pi\alpha c V_\infty$$

* Lift : $$L' = \rho_\infty V_\infty \Gamma = \pi\alpha c \rho_\infty V_\infty^2$$

* Lift coefficient : $$C_l = \frac{L'}{q_\infty c} = \frac{\pi\alpha c \rho_\infty V_\infty^2}{\frac{1}{2}\rho_\infty V_\infty^2 c} = 2\pi\alpha$$

* Lift slope : $$\frac{dC_l}{d\alpha} = 2\pi$$

➔ Lift coefficient is linearly proportional to angle of attack.

< 4.7 Classical Thin Airfoil Theory >

❖ The Symmetric Airfoil

* The moment about the leading edge

➔ $$M_{LE}' = -\int_0^c \xi (dL)$$

$$= -\rho_\infty V_\infty \int_0^c \xi \gamma(\xi) d\xi$$

$$= -\rho_\infty V_\infty \int_0^\pi \frac{c}{2}(1-\cos\theta) 2\alpha V_\infty \left(\frac{1+\cos\theta}{\sin\theta}\right)\frac{c}{2} sin\theta \, d\theta$$

$$= -\rho_\infty V_\infty^2 (2\alpha)\left(\frac{c}{2}\right)^2 \int_0^\pi (1-\cos^2\theta) d\theta$$

$$= -\frac{1}{2}\rho_\infty V_\infty^2 \alpha c^2 \int_0^\pi \sin^2\theta \, d\theta$$

$$= -q_\infty c^2 \frac{\pi\alpha}{2}$$

< 4.7 Classical Thin Airfoil Theory >

❖ The Symmetric Airfoil

* The moment coefficient

➔ $$C_{m,LE} = -\frac{M_{LE}'}{q_\infty S c} = \frac{M_{LE}'}{q_\infty c^2} = -\frac{\pi\alpha}{2}$$

$$C_l = 2\pi\alpha$$

$$C_{m,LE} = -\frac{C_l}{4}$$

$$C_{m,c/4} = C_{m,LE} + \frac{C_l}{4} = 0$$

* Aerodynamic center is located at c/4 for incompressible, inviscid and symmetric airfoil (true in real world)

* Center of pressure : the point at which the moment is zero
Aerodynamic center : the point at which the moment is independent of aoa

< 4.8 The Cambered Airfoil >

* From thin airfoil theory,

➔ $$\frac{1}{2\pi}\int_0^c \frac{\gamma(\xi)d\xi}{(x-\xi)} = V_\infty(\alpha - \frac{dz}{dx})$$ ...... (a)

* For cambered airfoil, $\frac{dz}{dx} \neq 0$

Transform ξ into θ ➔ $$\frac{1}{2\pi}\int_o^\pi \frac{\gamma(\theta)\sin\theta\, d\theta}{\cos\theta - \cos\theta_0} = V_\infty(\alpha - \frac{dz}{dx})$$ ...... (b)

* The solution becomes

$$\gamma(\theta) = 2V_\infty(A_0\frac{1+\cos\theta}{\sin\theta} + \sum_{n=1}^{\infty} A_n \sin n\theta)$$ ...... (c)

Leading term for symmetric airfoil

Fourier series term due to camber

< 4.8 The Cambered Airfoil >

* Substitute (c) into (b)

$$\rightarrow \frac{1}{\pi}\int_0^{\pi}\frac{A_0(1+\cos\theta)d\theta}{\cos\theta-\cos\theta_0}+\frac{1}{\pi}\int_0^{\pi}\frac{A_n\sin n\theta\sin\theta\, d\theta}{\cos\theta-\cos\theta_0}=\alpha-\frac{dz}{dx}$$

By using the integral standard form

$$\int_0^{\pi}\frac{\sin n\theta\sin\theta\, d\theta}{\cos\theta-\cos\theta_0}=-\pi\cos n\theta_0$$

$$\int_0^{\pi}\frac{\cos n\theta\, d\theta}{\cos\theta-\cos\theta_0}=\frac{\pi\sin n\theta_0}{\sin\theta_0}$$

$$\rightarrow \quad A_0-\sum_{n=1}^{\infty}A_n\cos n\theta_0=\alpha-\frac{dz}{dx}$$

$$\rightarrow \quad \frac{dz}{dx}=(\alpha-A_0)+\sum_{n=1}^{\infty}A_n\cos n\theta_0$$

< 4.8 The Cambered Airfoil >

For Fourier cosine series,

$$f(\theta) = B_0 + \sum_{n=1}^{\infty} B_n \cos n\theta, \;\; 0 \le \theta \le \pi$$

$$B_0 = \frac{1}{\pi}\int_0^{\pi} f(\theta)\, d\theta$$

$$B_n = \frac{2}{\pi}\int_0^{\pi} f(\theta)\cos n\theta \, d\theta$$

➔

$$A_0 = \alpha - \frac{1}{\pi}\int_0^{\pi} \frac{dz}{dx}\, d\theta_0$$

$$A_n = \frac{2}{\pi}\int_0^{\pi} \frac{dz}{dx} \cos n\theta_0 \, d\theta_0$$

[Note] given $\frac{dz}{dx}, \alpha$ **➔ Determine γ(θ) to make the camber line a streamline with** A_0**,** A_n
+ Kutta condition, γ(π)=0

< 4.8 The Cambered Airfoil >

* The total circulation due to the entire vortex sheet

$$\Gamma = \int_0^c \gamma(\xi)d\xi = \frac{c}{2}\int_0^\pi \gamma(\theta)\sin\theta\, d\theta \quad \leftarrow \gamma(\theta) = 2V_\infty\left(A_0\frac{1+\cos\theta}{\sin\theta} + \sum_{n=1}^{\infty} A_n \sin n\theta\right)$$

$$= cV_\infty\left[A_0\int_0^\pi (1+\cos\theta)d\theta + \sum_{n=1}^{\infty} A_n \int_0^\pi \sin n\theta \sin\theta\, d\theta\right]$$

By using $\int_0^\pi (1+\cos\theta)d\theta = \pi$, $\int_0^\pi \sin n\theta \sin\theta\, d\theta = \begin{cases} \frac{\pi}{2} & \text{for } n = 1 \\ 0 & \text{for } n \neq 1 \end{cases}$

➔ $\Gamma = cV_\infty\left(\pi A_0 + \frac{\pi}{2}A_1\right)$

< 4.8 The Cambered Airfoil >

* Lift coefficient for a cambered thin airfoil

➔ $$L' = \rho_\infty V_\infty \Gamma = \rho_\infty V_\infty^2 c\left(\pi A_0 + \frac{\pi}{2} A_1\right)$$

➔ $$C_l = \frac{L'}{\frac{1}{2}\rho_\infty V_\infty^2 c} = \pi(2A_0 + A_1)$$

$$\begin{cases} A_0 = \alpha - \dfrac{1}{\pi}\displaystyle\int_0^\pi \frac{dz}{dx} d\theta_0 \\ A_1 = \dfrac{2}{\pi}\displaystyle\int_0^\pi \frac{dz}{dx} \cos\theta_0 \, d\theta_0 \end{cases}$$

➔ $$C_l = 2\pi\left(\alpha + \frac{1}{\pi}\int_0^\pi \frac{dz}{dx}(\cos\theta_0 - 1)\, d\theta_0\right)$$

➔ **Lift slope,** $a_0 = \dfrac{dC_l}{d\alpha} = 2\pi$

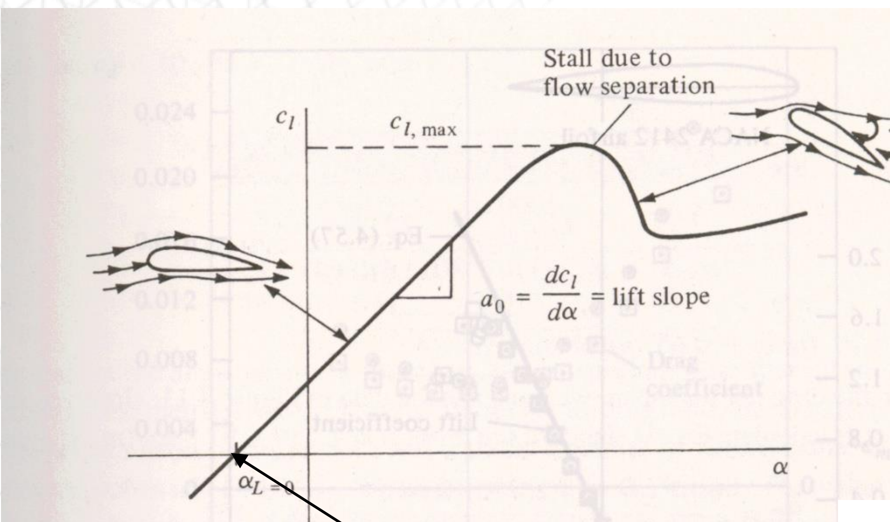
< 4.8 The Cambered Airfoil >

* Lift coefficient for a cambered thin airfoil

$$C_l = 2\pi\left(\alpha + \frac{1}{\pi}\int_0^{\pi} \frac{dz}{dx}(\cos\theta_0 - 1)\,d\theta_0\right)$$

[Note]

* Lift slope is 2π for any shape airfoil



$$c_l = \frac{dc_l}{d\alpha}(\alpha - \alpha_{L=0}) = 2\pi(\alpha - \alpha_{L=0})$$

* Zero lift angle : $\alpha_{L=0} = -\frac{1}{\pi}\int_0^{\pi} \frac{dz}{dx}(\cos\theta_0 - 1)\,d\theta_0$

< 4.8 The Cambered Airfoil >

* The total moment about the leading edge

$$M_{LE} = -\int_0^c \xi(dL) = -\int_0^c \rho_\infty V_\infty \gamma(\xi) d\xi$$

$$= -\rho_\infty V_\infty \int_0^\pi 2V_\infty \left(A_0 \frac{1+\cos\theta}{\sin\theta} + \sum_{n=1}^{\infty} A_n \sin n\theta\right) \frac{c}{2}(1-\cos\theta)\left(\frac{c}{2} sin\theta\right) d\theta$$

$$= -\frac{1}{2}\rho_\infty V_\infty^2 c^2 \left[\int_0^\pi A_0(1-\cos^2\theta) d\theta + \int_0^\pi A_n \sin n\theta \sin\theta d\theta - \int_0^\pi A_n \sin n\theta \sin\theta \cos\theta d\theta\right]$$

$$= -\frac{1}{2}\rho_\infty V_\infty^2 c^2 \left[\frac{\pi}{2}\left(A_0 + A_1 - \frac{A_2}{2}\right)\right]$$

* Moment coefficient

$$C_{m,LE} = \frac{M_{LE}}{q_\infty c^2} = -\frac{\pi}{2}\left(A_0 + A_1 - \frac{A_2}{2}\right) \quad \leftarrow \quad C_l = \pi(2A_0 + A_1)$$

$$= -\left[\frac{C_l}{4} + \frac{\pi}{4}(A_1 - A_2)\right]$$

➔ $C_{m,\frac{c}{4}} = \frac{\pi}{4}(A_2 - A_1)$

➔ A_1 & A_2 both are independent of aoa
➔ The quarter-chord is the aerodynamic center for a cambered airfoil

< 4.8 The Cambered Airfoil >

* The center of pressure

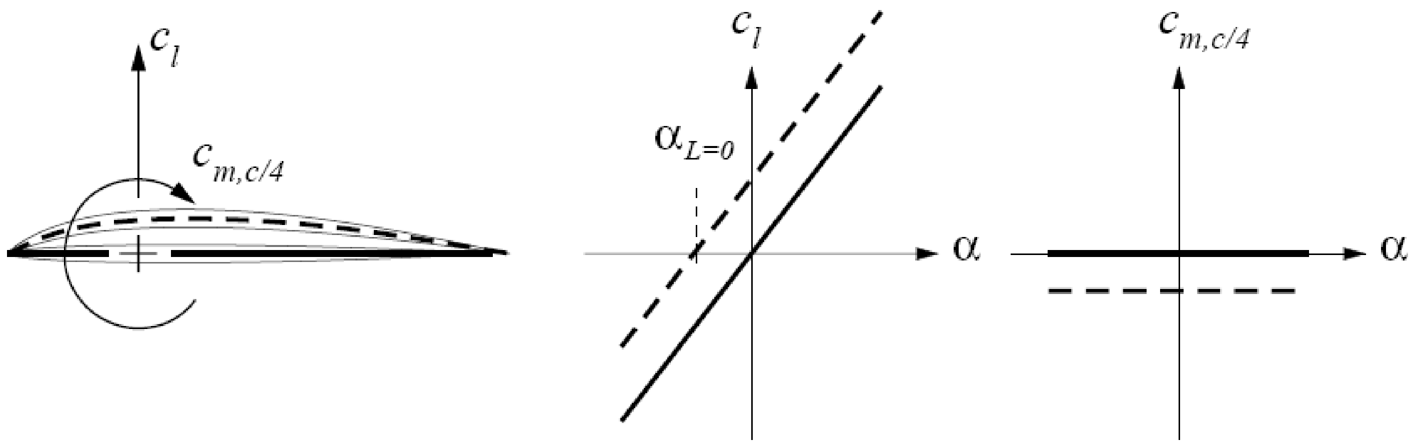
$$x_{cp} = -\frac{M_{LE}}{L} = -\frac{C_{m,LE}c}{C_l}$$

$$= \frac{\left[\frac{C_l}{4} + \frac{\pi}{4}(A_1 - A_2)\right]c}{C_l}$$

$$= \frac{c}{4}\left[1 + \frac{\pi}{C_l}(A_1 - A_2)\right]$$ ➔ $f(\alpha)$ ➔ Not a convenient point

< 4.8 The Cambered Airfoil >

❖ The influence of camber on the thin airfoil



- - - * The cambered airfoil

$$C_l = 2\pi\left(\alpha + \frac{1}{\pi}\int_0^{\pi}\frac{dz}{dx}(\cos\theta_0 - 1)\,d\theta_0\right)$$

$$C_{m,\frac{c}{4}} = \frac{\pi}{4}(A_2 - A_1)$$

—— * The symmetric airfoil

$$C_l = 2\pi\alpha$$

$$C_{m,c/4} = C_{m,LE} + \frac{C_l}{4} = 0$$